Exactly solvable sandpile with fractal avalanching

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A simple one-dimensional sandpile model is constructed which possesses exact analytical solvability while displaying both scale-free behavior and fractal properties. The sandpile grows by avalanching on all scales, yet its shape and energy content are described by a simple, continuous (but nowhere differentiable) analytical formula. The avalanche energy distribution and the avalanche time series are both power laws with index -1 (''$1/f$ spectra''). [S1063-651X(99)00306-2]

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In this paper, we describe a very simple one-dimensional sandpile model which possesses exact analytical solvability while displaying both fractal properties and a behavior of a type sometimes identified as self-organized criticality (SOC) [1,2]. In contrast to most sandpile algorithms in the literature, the model described in the present paper is completely deterministic. Neither the fueling nor the redistribution of material within the pile contains any randomness. Our objective is to show that, even though the growth of the sandpile can be regarded as a manifestation of SOC, its detailed behavior can be understood analytically. The sandpile grows in a self-similar—indeed fractal—manner, and the time evolution of its shape and energy storage is described by a simple, continuous (but nowhere differentiable) analytical formula which we derive.

The model represents a sandpile fueled, one grain at a time, at its first node, $n=1$. The number of nodes is semi-infinite, $1 \leqslant n < \infty$, so that the sandpile can build and spread indefinitely by avalanching as the fueling continues. The model shares many standard features of sandpile algorithms: when the height difference between two neighboring nodes gives rise to a local gradient $z_n = h_n - h_{n+1}$ that exceeds a critical gradient z_c, this triggers a local redistribution of sand to flatten the gradient at the node n. This redistribution may then steepen the slope at nearby nodes, triggering further redistribution. Thus the local critical gradient condition, plus local redistribution, can give rise to global avalanches. The redistribution is regarded as instantaneous, so that the next grain is added only when the redistribution is complete, with $z_n < z_c$ for all n. In our model, we set the final gradient at each node that has participated in an avalanche to the angle of repose z_r, which is smaller than z_c. In other words, if an avalanche occurs in a given time step, *all* nodes that participate in that avalanche then relax to $z_n = z_r$. In this respect, the algorithm differs from the standard sandpile models in the literature [1,2].

To understand how the algorithm works in detail, suppose a certain node n becomes unstable because of the addition of a grain of sand. As in conventional sandpile algorithms, some material then falls down to the next position, so that the slope z_n is reduced to the angle of repose z_r. As the sand below the angle of repose plays no part in the sandpile dynamics, we can take $z_r = 0$, and refer to this state as ''flat.'' Thus the first step in the redistribution process is

$$h_n^* = h_{n+1}^* = (h_n + h_{n+1})/2,$$

where an asterisk denotes values after redistribution. The slopes then become

$$z_{n-1}^* = z_{n-1} + z_n/2,$$

$$z_n^* = 0,$$

$$z_{n+1}^* = z_n/2 + z_{n+1},$$

as in conventional sandpile algorithms. However, if either of the neighboring nodes, $n-1$ or $n+1$, now becomes unstable, the behavior of our model differs from most algorithms in the literature. For instance, if the node $n+1$ is unstable, the subsequent redistribution occurs in such a way as to flatten the *entire* unstable region $[n, n+1]$ while conserving the total number of sand grains, i.e., the heights become

$$h_n^* = h_{n+1}^* = h_{n+2}^* = (h_n + h_{n+1} + h_{n+2})/3,$$

and the slopes

$$z_{n-1}^* = z_{n-1} + 2z_n/3 + z_{n+1}/3,$$

$$z_n^* = z_{n+1}^* = 0,$$

$$z_{n+2}^* = z_n/3 + 2z_{n+1}/3 + z_{n+2}.$$

If this makes either of the nodes $n-1$ and $n+2$ unstable, the redistribution continues in a similar manner until the pile has relaxed to a completely stable state. The relaxation is assumed to occur on a time scale faster than that of external fueling, so that no sand is added to the pile until it has relaxed to a stable state. Note that the amount of sand in the pile is conserved in the redistribution process.

The present model can be viewed as a special case of a more general sandpile algorithm that we have described earlier [3]. The latter includes a free parameter y that permits a probabilistic spread in the value of z_c, and yields interesting results with potentially wide physical applications [4–6].

The sandpile considered in the present paper is equivalent to that of Ref. [3] in the particularly simple limit where the critical gradient is sharply defined (the limit $y \to \infty$).

The fueling is assumed to occur only at the node $n=1$. Thus neither the redistribution nor the fueling contains any randomness, and the system is completely deterministic. The dynamics of the sandpile is then analytically tractable and can be deduced from two basic considerations: (i) Each avalanche begins at the first node of the pile, $n=1$. (ii) The pile can only conquer new ground, i.e., spread to previously unoccupied positions at large n, by an avalanche involving the last position in the existing pile. Immediately after an avalanche, the pile is completely flat at all previously unstable nodes. Thus after an avalanche involving the nodes $1 \leqslant n < N$, all these nodes have $z_n = 0$, and z_N is increased by the amount necessary to keep constant the total mass of the sandpile,

$$\sum_{n=1}^{\infty} h_n = \sum_{n=1}^{\infty} \sum_{j=n}^{\infty} z_j = \sum_{n=1}^{\infty} n z_n .$$

In this case the redistribution rules imply

$$z_n^* = 0, \quad 0 \leqslant n < N,$$

$$z_N^* = \frac{1}{N} \sum_{n=1}^{N} n z_n ,$$

where N is chosen as the smallest integer for which this new z_N does not exceed z_c. Note that the division by N implies that z_N cannot be constrained to be an integer.

In order for the pile to spread horizontally, the entire pile must be involved in an avalanche. Suppose that the number of occupied nodes (the ‘‘width’’ of the pile) immediately after an avalanche that completely flattens the pile is equal to $N \gg 1$, and that this number is considerably smaller than the maximum allowed width of the pile (so that there is no interaction with the edge). Since the pile is flat in the region $1 \leqslant n < N$, the heights at all these positions are identical, and we normalize them to unity at this time: $h_n = 1$ for $1 \leqslant n < N$ and $h_n = 0$ for $n > N$. The total amount of sand in the pile is then equal to N, and the energy of the pile is

$$E_N^+ = \sum_{1}^{N} h_n^2 = N.$$

The pile is assumed to be fueled by grains of sand much smaller than unity, and for convenience we also normalize the unit of time to the rate of fueling, so that one unit of sand is added in unit time. (This corresponds to a large number of grains.) Thus the avalanche that we have just described occurs at the time $t=N$ after the fueling began.

When the fueling at $n=1$ commences again at $t>N$, the newly added sand builds a pile on top of the already existing, flattened pile. As long as no subsequent avalanche reaches the position $n=N$, the dynamics of this buildup is exactly the same as that before the avalanche, since the redistribution algorithm is independent of the absolute height. (The redistribution rules are unaffected by a transformation $h_n \to h_n + \text{const.}$) In other words, a pile identical to that before the system-wide avalanche builds up across the upper surface of

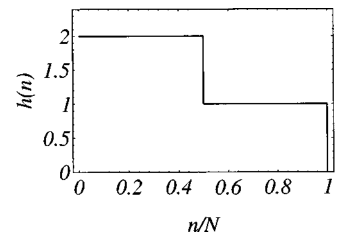

FIG. 1. The shape of the sandpile just after the avalanche at $t = 3N/2$.

the flattened pile. At the time $t=2N$, this pile experiences an avalanche which is at first identical to that which occurred at $t=N$, but when it reaches the node at $n=N$, this node now becomes unstable. The underlying material at $n=N$ then starts participating in the avalanche, which does not stop until the pile has spread to $n=2N$, because the amount of sand in the pile is now exactly twice that in the system at $t=N$, and the final state is flattened everywhere. The energy after the event at $t=2N$ is therefore

$$E_{2N}^+ = 2N.$$

Now let us follow this chain of thought backwards in time. It is clear that an avalanche involving the entire pile doubles its width (the number of positions with $h_n > 0$). Immediately before $t=N$, the pile therefore had a width of $N/2$, and immediately before $t=N/2$ its width was $N/4$, and so on. As has already been described, during the time interval $N<t<2N$, a pile the same as that created during $0<t<N$ builds up on top of the flattened pile created at $t=N$, for which $h_n=1$, $1 \leqslant n < N$. By analogy, it then follows that immediately after $t=3N/2$, the sandpile consists of a flat ‘‘ground floor’’ of width N, upon which rests a smaller flat pile of width $N/2$ and unit height (similar to that which existed at $t=N/2$)—see Fig. 1. Further addition of sand beyond $t=3N/2$ builds on top of this second, or ‘‘first floor,’’ story.

The process iterates in a self-similar manner, and it is clear that just before the system-wide avalanche at $t=2N$, the pile is as shown in Fig. 2. It consists of a flat, ground floor of length N; on top of that there is a first floor of length $N/2$; on top of this rests a second floor of length $N/4$; and so on. All these stories are of unit height. Analogously, just before $t=N$, the sandpile had the same shape, but with all horizontal dimensions scaled down by a factor of 2.

To describe the shape of the sandpile at arbitrary t, it is useful to pass to the continuum limit, where N is very large and the horizontal coordinate along the pile (x) can be defined as a continuous variable replacing the former discrete

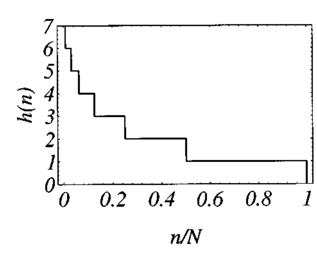

FIG. 2. The shape of the sandpile just before the system-wide avalanche at $t=2N$.

node label n. We scale x so that the pile occupies the interval $0<x<1$ just after $t=N$, and introduce a normalized unit of time s, such that $s=1$ when $t=N$. The shape of the sandpile can then be related to the binary representation of s in a simple way. The instant just after $t=N$ corresponds to $s=1.0$, while the instant just before $t=N$ corresponds to $s=0.111\ldots$. If we let $h(x,s)$ represent the height of the sandpile, we have

$$h(x,1.0)=1, \quad 0<x<1,$$

while at $s=0.111\ldots$ the pile resembles an infinitely high staircase with increasingly narrow steps, as in Fig. 2, and its bottom step extending to $x=\frac{1}{2}$:

$$h(x,0.111\ldots)=n, \quad 2^{-(n+1)}<x<2^{-n} \quad (0\leqslant n\leqslant\infty).$$

By the same argument we may use the binary series expression for an arbitrary time,

$$s=\sum_{n=-\infty}^{\infty} s_n 2^{-n},$$

where $s_n=0$ for all n smaller than some finite integer, to write the shape of the pile as

$$h(x,s)=\sum_{j\leqslant n} s_j, \quad 2^{-(n+1)}<x<2^{-n}. \tag{1}$$

We are now in a position to calculate the normalized energy of the pile as a function of time,

$$E(s)=\int_0^\infty h^2(x,s)dx,$$

by summing the contributions from each horizontal segment $2^{-(n+1)}<x<2^{-n}$. Since the width of such a segment is $2^{-(n+1)}$, the total energy of the pile is

$$E(s)=\sum_{n<\infty} 2^{-(n+1)}\left(\sum_{j\leqslant n} s_j\right)^2.$$

By writing this sum as

$$E(s)=\sum_{n<\infty} 2^{-(n+1)}\sum_{j\leqslant n} s_j\left(s_j+2\sum_{k<j} s_k\right)$$
$$=\sum_{j<\infty} s_j\sum_{n=j}^{\infty} 2^{-(n+1)}\left(s_j+2\sum_{k<j} s_k\right),$$

and noting that $s_j^2=s_j$ since $s_j=0$ or 1, the summations with respect to n can be evaluated, giving

$$E(s)=s+2\sum_{j<\infty} 2^{-j}s_j\sum_{j>k} s_k.$$

Finally, interchanging the summation with respect to j and k, and defining $[s]_k$ as the value of s when truncated to k binary places,

$$[s]_k=\sum_{j\leqslant k} s_j 2^{-j},$$

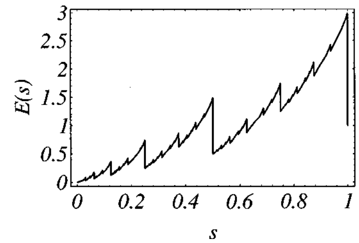

FIG. 3. The energy of the sandpile vs normalized time [Eq. (2)].

we can write the energy as

$$E(s)=s+2\sum_{k<\infty} s_k(s-[s]_k). \tag{2}$$

The first term describes the average, linear increase of energy with time, and the second the fluctuations, including avalanches, which occur at every instant of time. Indeed, an avalanche occurs whenever a digit s_k in the binary representation of s changes, and the size of the avalanche is related to k. Large avalanches are associated with small k, small ones with large k. Figure 3 shows a plot of energy versus time. Note that it is a continuous, but nowhere differentiable, function.

Let us now determine the distribution function of avalanche sizes. It follows from Eq. (2) that the energy released in the avalanche at $s=\frac{1}{2}$ is equal to unity,

$$\Delta E(\tfrac{1}{2})=E(0.01111\ldots)-E(0.1000\ldots)=1.$$

It follows from the self-similarity argument above, and also from Eq. (2), that the avalanche at $s=\frac{1}{4}$ dissipates half of this energy, $\Delta E(\frac{1}{4})=\frac{1}{2}$. On the other hand, the avalanche at $s=\frac{3}{4}$ is exactly as large as that at $s=\frac{1}{4}$ since the only difference is that they cause spreading across ''floors'' at different heights: the avalanche at $s=\frac{1}{4}$ spreads the sandpile on the ground, whereas that at $s=\frac{3}{4}$ occurs on the ''first floor'' but is otherwise the same. Figure 4 shows the energy released in avalanches, ΔE, as a function of time, and it is clear that

$$\Delta E(s)=2^{1-n},$$

where n is the position of the leftmost digit in the binary representation of s that changes at the time of the avalanche, i.e., the smallest integer such that $s_k=1$ for all $k>n$. It also follows that avalanches of any particular size are exactly twice as common as those of twice that size. The probability distribution of avalanche sizes is thus an inverse power law distribution,

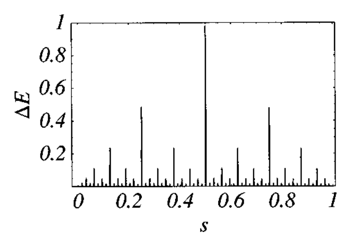

FIG. 4. Energy release in avalanches vs normalized time.

$$P(\Delta E)\propto 1/\Delta E.$$

Since in our model the sandpile grows indefinitely across an infinite ''table,'' it never reaches a steady state. If the system was finite, with an open boundary at $x=L$, the behavior would be periodic with a period equal to the smallest power of 2 larger than L. For example, suppose $2^{N-1}<L<2^N$; for $s<2^N$ the system would behave as described above, while at $s=2^N$ a major avalanche would completely empty the sandpile. Hence the period would be 2^N. It is worth noting that the energy dissipated in this avalanche is $\Delta E=2^{N+1}$, and thus scales linearly, rather than quadratically, with the system size L. This has do to with the fact that the average height of the sandpile does not increase with L. (Half the time the height of the pile is 1, one quarter of the time it is 2, and so on, independently of L.) As the mass $\int h(x,t)dx$ of the pile increases linearly with time, the mass leaving the system in each system-emptying avalanche is equal to the period, which is 2^N and thus scales linearly with L.

Not only the energy distribution of the internal avalanches, but also the Fourier spectrum in the time domain (the coefficients of the Fourier series) of the avalanches in the finite sandpile, is an inverse power law. To see this, we take a finite sandpile of the type we have just discussed, of length $\frac{1}{2}<L<1$, so that the period is equal to unity. Consider the infinite series of δ functions representing the energy released in the sequence of avalanches shown in Fig. 4,

$$q(s)=2\,\delta(s)+\delta(s-1/2)+\tfrac{1}{2}[\delta(s-1/4)+\delta(s-3/4)] + \tfrac{1}{4}[\delta(s-1/8)+\delta(s-3/8)+\delta(s-5/8)+\delta(s-7/8)] +\cdots.$$

The Fourier series representation of this periodic function has coefficients

$$c_n=\int_0^1 q(s)e^{in\pi s}\,ds,$$

By rearranging the terms in the expression for $q(s)$, we obtain

$$c_n=1+\tfrac{1}{2}(1+e^{in\pi/2})+\tfrac{1}{4}(1+e^{in\pi/4}+e^{2in\pi/4}+e^{3in\pi/4})+\cdots=\sum_{N=0}^{\infty}2^{-N}\sum_{k=0}^{2^N-1}\exp(ikn\pi 2^{-N}).$$

It follows from this expression that c_n vanishes if n is even, and is equal to

$$c_n=\sum_N \frac{2^{1-N}}{1-\exp(in\pi 2^{-N})}$$

for odd n. Since

$$\lim_{N\to\infty}\frac{2^{1-N}}{1-\exp(in\pi 2^{-N})}=\frac{2i}{n\pi},$$

this sum does not converge, and, strictly speaking, the Fourier series representation does not exist in the continuum limit we are considering where the sand grains are infinitesimally small. However, if the grains added to the pile have a small but finite size, the sum should be truncated at some large but finite N corresponding to this size, and it follows from the above that $c_n\propto 1/n$. In other words, the Fourier coefficients of the finite-sandpile avalanche energy signal follow a power law with index -1 (a ''$1/f$ spectrum'') in the time domain.

Most sandpile models in the literature are difficult to analyze analytically, so that much of the present understanding of sandpile models is derived from numerical simulations. In this paper, we have constructed a simple mathematical model of a sandpile that shares many features of the conventional models, such as external fueling, uniform and invariant critical gradient, local redistribution, and post-avalanche flattening at participating nodes. The pile grows by fractal avalanching on all scales, and yet is completely analytically tractable. This result sheds some light on the links between sandpile models [7,8] self-organized criticality [9–11] predictability, fractality, and determinism [12–14]. Equation (1) above gives the shape of the sandpile at any time, and Eq. (2) represents its energy. The latter yields a time series that combines fractal and scale-free properties, sometimes taken as evidence for SOC. For instance, both the avalanche energy distribution and the finite-sandpile avalanche time series have $1/f$ spectra. We have shown that this system can be understood completely without implementing a numerical cellular automaton algorithm.

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